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EXISTENCE OF UPPER SOLUTION OF FIE INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract: In this paper, we will extend to prove the existence of maximal solution of quadratic fractional integral equation involving the generalized Mittag-Leffler function and this maximal solution will serve as an upper bound for the solution and this solution we got with the help of approximation of the integral equation by sequence of solution converging to this.

Keywords and Phrases: Quadratic fractional integral equation, Fractional derivatives and Integrals, Approximate solution.

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1. Introduction

Quadratic integral equations was investigated by many authors since long time because of their useful applications in describing numerous events and problems of the real world. Initial studies by Chandrasekhar [11, 12] form only a beginning for this theory, mainly made by astrophysicists. After observing the occurrence in the problems of some natural and physical processes of the universe, for example in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory, Especially, the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications [8, 9, 10, 11, 14, 15, 26]. Research was conducted by mathematicians. They found some interesting open questions in this theory, some of them are Anichini, Conti, Cahlon, Eskin, Banas, Argyros, Caballero, Mingarelli, Sadarangani, Nussbaum, Gripenberg, Mullikin, Rus, Shrikhant, Joshi, Schillings and many others. The existence results for such quadratic operators equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities involved in the quadratic operator or functional equations. The hybrid fixed point theorems in Banach algebras find numerous applications in the theory of nonlinear quadratic differential and integral equations [16, 17, 18, 21, 22]. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper, we prove the existence as well as approximations of the solutions of a certain generalized quadratic integral equation via an algorithm based on successive approximations under weak partial Lipschitz and compactness type conditions.

The existing work for quadratic integral equations does not guarantee the approximation of the solution. In this paper the approximation form by the algorithm guarantee the solution of non-linear quadratic fractional differential equations involving generalized Mittag-Leffler function.

Given a closed and bounded interval $J = [0, T]$ of the real line R for some $T > 0$, we consider the quadratic fractional integral equation (in short QFIE)

$$x(t) = f(t, x(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q) g(s, x(s)) ds \right) \tag{1.1}$$

where $f, g : J \times \mathbb{R} \to \mathbb{R}$ and $q : J \to \mathbb{R}$ are continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function, and $E_{\alpha,\beta}^{\gamma,\delta,q}(x)$ is generalized Mittag-Leffler function.

By a solution of the QFIE (1.1), we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is **regular** if $\{x_n\}_{n\in\mathbb{N}}$ is a non-decreasing (resp. non-increasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [24] and

the references therein.
In this section, we present some basic definitions and preliminaries which are useful in further discussion.

Definition 2.1. (Mittag-Leffler Function) [30] *The Mittag - Leffler function of one parameter is denoted by* $E_\alpha(z)$ *and defined as,*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{2.1}$$

where $z, \alpha \in \mathbb{C},\ Re(\alpha) > 0.$
If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{2.2}$$

Definition 2.2. (Mittag-Leffler Function for two parameters) *The generalization of* $E_\alpha(z)$ *was studied by Wiman (1905)* [35], *Agarwal* [7] *and Humbert and Agarwal* [27] *defined the function as,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k \tag{2.3}$$

where $z, \alpha, \beta \in \mathbb{C},\ Re(\alpha) > 0, Re(\beta) > 0,$
In 1971, The more generalized function is introduced by Prajapati [34] as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)} \tag{2.4}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C},\ Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,$
where $(\gamma \neq 0, \gamma)_k = \gamma(\gamma+1)(\gamma+2)...(\gamma+k-1)$ is the Pochhammer symbol [34], and

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}$$

In 2007, Shulka and Prajapati [34] introduced the function which is defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k!\Gamma(\alpha k + \beta)} \tag{2.5}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}$, $min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \cup N$.
In 2012, further generalization of Mittag - Leffler function was defined by Salim [33] and Chauhan [13] as,

$$E_{\alpha,\beta}^{\gamma,\delta,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_(qk)\Gamma(\alpha k + \beta)}. \tag{2.6}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}$, $min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \cup N$

$$(\gamma)_{qk} = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)} \quad \text{and} \quad (\delta)_{qk} = \frac{\Gamma(\delta + qk)}{\Gamma(\delta)}$$

denote the generalized Pochhammer symbol [34],

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* ***isotone*** *or* ***non-decreasing*** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.*

Definition 2.4. [19] *A mapping $\mathcal{T} : E \to E$ is called* ***partially continuous*** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain c contained in E.*

Definition 2.5. *A mapping $\mathcal{T} : E \to E$ is called* ***partially bounded*** *if $\mathcal{T}(c)$ is bounded for every chain c in E. $\mathcal{T}$ is called* ***uniformly partially bounded*** *if all chains $\mathcal{T}(c)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* ***bounded*** *if $\mathcal{T}(E)$ is a bounded subset of E.*

Definition 2.6. *A mapping $\mathcal{T} : E \to E$ is called* ***partially compact*** *if $\mathcal{T}(c)$ is a relatively compact subset of E for all totally ordered sets or chains c in E. $\mathcal{T}$ is called* ***uniformly partially compact*** *if $\mathcal{T}(c)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* ***partially totally bounded*** *if for any totally ordered and bounded subset c of E, $\mathcal{T}(c)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* ***partially completely continuous*** *on E.*

Definition 2.7. [19] *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone non-decreasing or monotone non-increasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric*

d defined through the norm $\|\cdot\|$ are compatible.

Definition 2.8. [16] *A upper semi-continuous and monotone non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.7}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } \ u, v \in E^+\}. \tag{2.8}$$

The elements of $\mathcal{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.1. [17] *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.9. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

Theorem 2.2. [20] *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra, such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two non-decreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$,

(b) $\mathcal{B}$ is partially continuous and uniformly partially compact, and

(c) $M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : c$ is a chain in $E\}$ and

(d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x = x \tag{2.9}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n,\ n = 0, 1, \ldots,$ converges monotonically to x^*.

3. Main Result

The QFIE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.2}$$

for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t.the above partially order relation $\leq$.

The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively.Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Definition 3.1. *A function $v \in C(J, \mathbb{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies*

$$v(t) \leq f(t, v(t)) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q) g(s, v(s)) ds\right)$$

for all $t \in J$. Similarly, a function $u \in C(J, \mathbb{R})$ is said to be an upper solution of the QFIE (1.1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_1) The functions $f, g : J \times \mathbb{R} \to \mathbb{R}_+, q : J \to \mathbb{R}_+$ where q is continuous function.

(A_2) There exists constants $M_f, M_g > 0$ such that $0 \leq f(t, x) \leq M_f$ and $0 \leq g(t, x) \leq M_g$ for all $t \in J$ and $x \in \mathbb{R}$.

(A_3) There exists a $\mathcal{D}$-function ψ_f such that

$$0 \leq f(t, x) - f(t, y) \leq \psi_f(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$,$x \leq y$.

(A_4) $g(t,x)$ is non-decreasing in x for all $t \in J$.

(A_5) The QFIE (1.1) has a lower solution $v \in C(J,\mathbb{R})$.

Theorem 3.2. [31] *Assume that hypotheses (A_1)-(A_5) holds then the QFIE (1.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by*

$$x_{n+1}(t) = \left[f(t,x_n(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\; g(s,x_n(s))\,ds\right) \tag{3.3}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

Definition 3.2. *A function $r \in C(J,\mathbb{R})$ is said be a maximal solution of the QFIE (1.1) if for any other solution x of the QFIE (1.1), one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a minimal solution ρ of the QFIE (1.1) can be defined in a similar way by reversing the above inequality.*
The following lemma is fundamental in the proof of maximal and minimal solutions for the QFIE (1.1) on J.

Lemma 3.3. *Suppose that there exist two functions $y, z \in C(J,\mathbb{R})$ satisfying*

$$y(t) \leq \left[f(t,y(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\; g(s,y(s))\,ds\right) \tag{3.4}$$

and

$$z(t) \geq \left[f(t,z(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\; g(s,z(s))\,ds\right) \tag{3.5}$$

for all $t \in J$. If one of the inequalities (3.4) and (3.5) is strict, then

$$y(t) < z(t) \tag{3.6}$$

for all $t \in J$.
Proof. Suppose that the inequality (3.5) is strict and let the conclusion (3.6) be false. Then there exists $t_1 \in J$ such that

$$y(t_1) = z(t_1), t_1 > 0,$$

and

$$y(t) < z(t), 0 < t < t_1.$$

From the monotonicity of $f(t,x)$ and $g(t,x)$in x , we get

$$\begin{aligned} y(t_1) &\leq \left[f(t_1,y(t_1))\right]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g(s,y(s))\,ds\right)\\ &= \left[f(t_1,z(t_1))\right]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g(s,z(s))\,ds\right)\\ &< z(t_1) \end{aligned} \tag{3.7}$$

which contradicts the fact that $y(t_1)=z(t_1)$. Hence, $y(t)<z(t)$ for all $t\in J$.

Theorem 3.4. *Suppose that all the hypotheses of theorem 3.2 hold. Then the QFIE (1.1) has a maximal and a minimal solution on J.*

Proof. Let $\epsilon>0$ be given. Now consider the fractional integral equation

$$x_\epsilon(t)=\left[f_\epsilon(t,x_\epsilon(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ g_\epsilon(s,x_\epsilon(s))\,ds\right) \tag{3.8}$$

for all $t\in J$, where

$$f_\epsilon(t,x_\epsilon(t))=f(t,x_\epsilon(t))+\epsilon$$

and

$$g_\epsilon(t,x_\epsilon(t))=g(t,x_\epsilon(t))+\epsilon$$

Clearly, the function $f_\epsilon(t,x_\epsilon(t))$ and $g_\epsilon(t,x_\epsilon(t))$, satisfy all the hypotheses (A_1)-(A_5)and therefore, by theorem 3.2, QFIE (1.1) has at least a solution $x_\epsilon(t)\in C(J,\mathbb{R})$.

Let ϵ_1 and ϵ_2 be two real numbers such that $0<\epsilon_2<\epsilon_1<\epsilon$. Then, we have

$$x_{\epsilon_2}(t)=\left[f_{\epsilon_2}(t,x_{\epsilon_2}(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ g_{\epsilon_2}(s,x_{\epsilon_2}(s))\,ds\right)$$

$$x_{\epsilon_2}(t)=\left[f(t,x_{\epsilon_2}(t))+\epsilon_2\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s,x_{\epsilon_2}(s))+\epsilon_2)\,ds\right) \tag{3.9}$$

$$\begin{aligned} x_{\epsilon_1}(t) &= \left[f_{\epsilon_1}(t,x_{\epsilon_1}(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ g_{\epsilon_1}(s,x_{\epsilon_1}(s))\,ds\right)\\ &= \left[f(t,x_{\epsilon_1}(t))+\epsilon_1\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s,x_{\epsilon_1}(s))+\epsilon_1)\,ds\right)\\ x_{\epsilon_1}(t) &> \left[f(t,x_{\epsilon_2}(t))+\epsilon_2\right]\left(\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)(g(s,x_{\epsilon_2}(s))+\epsilon_2)\,ds\right) \end{aligned} \tag{3.10}$$

for all $t \in J$. Now, applying the Lemma 3.3 to the inequalities (3.9) and (3.10), we obtain

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \tag{3.11}$$

for all $t \in J$. Let $\epsilon_0 = \epsilon$ and define a decreasing sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then in view of the above facts $\{x_{\epsilon_n}\}$ is a decreasing sequence of functions in $C(J,\mathbb{R})$. We show that this is uniformly bounded and equicontinuous. Now, by hypotheses,

$$\begin{aligned}|x_{\epsilon_n}(t)| &\leq |\left[f_{\epsilon_n}(t, x_{\epsilon_n}(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right)| \\ &\leq r\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|x_{\epsilon_n}\| \leq r$ for all $n \in \mathbb{N}$. This shows that the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded. Next, we show that $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| &\leq |\left[f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1))\right] \times \\
&\quad \left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \\
&\quad - \left[f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))\right] \left(\frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right)| \\
&= |f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1)) - f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \\
&\quad \times |\left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \\
&\quad + |f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_1-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds \right. \\
&\quad \left. - \int_0^{t_2} (t_2-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds \right| \\
&\leq |f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1)) - f_{\epsilon_n}(t_2, x_{\epsilon_n}(t_2))| \\
&\quad \times |\left(\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\ g(s, x_{\epsilon_n}(s))\, ds\right) \\
&\quad + |f(t_2, x_{\epsilon_n}(t_2))| \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_2-s)^q) g(s, x_n(s)) ds \right. \\
&\qquad \left. - \int_0^{t_2} (t_2-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q) g(s, x_n(s)) ds \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^{t_2}(t_2-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)g(s,x_n(s))ds\right.\\
&\qquad\left.-\int_0^{t_1}(t_2-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)g(s,x_n(s))ds\right|\\
&+\left|\int_0^{t_1}(t_2-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)g(s,x_n(s))ds\right.\\
&\qquad\left.-\int_0^{t_1}(t_1-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)g(s,x_n(s))ds\right|\\
&\le |f(t_1,x_{\epsilon_n}(t_1))-f(t_2,x_{\epsilon_n}(t_2))|\\
&\times|\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\ M_g ds\right)\\
&+M_f\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta}((t_2-s)^q)-E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\right||g(s,x_n(s))|\,ds\\
&+\left|\int_{t_1}^{t_2}(t_2-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)g(s,x_n(s))ds\right|\\
&+\int_0^{t_1}\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\,|g(s,x_n(s))|\,ds\\
&\le |f(t_1,x_{\epsilon_n}(t_1))-f(t_2,x_{\epsilon_n}(t_2))|\\
&\times|\left(\frac{1}{\Gamma(q)}\int_0^{T}(t_1-s)^{q-1}E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\ M_g ds\right)\\
&+M_f\frac{1}{\Gamma(q)}\int_0^{T}(t_2-s)^{q-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta}((t_2-s)^q)-E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\right|M_g ds\\
&+\int_0^{T}\left|(t_2-s)^{q-1}-(t_1-s)^{\alpha-1}\right|E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)M_g ds\\
&+\int_0^{T}\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)M_g ds
\end{aligned}
$$

$$
\begin{aligned}
|x_{\epsilon_n}(t_1)-x_{\epsilon_n}(t_2)| &\le |f(t_1,x_{\epsilon_n}(t_1))-f(t_2,x_{\epsilon_n}(t_2))|\\
&\times\left(\frac{1}{\Gamma(q)}\left(\int_0^T\left|(t_1-s)^{q-1}\right|^2ds\right)^{1/2}\left(\int_0^T\left|E^{\gamma,\delta,q}_{\alpha,\beta}((t_1-s)^q)\right|^2ds\right)^{1/2}\right)\\
&+M_f\frac{1}{\Gamma(q)}M_g\left(\int_0^T\left|(t_2-s)^{q-1}\right|^2ds\right)^{1/2}
\end{aligned}
$$

$$\left(\int_0^T \left|E_{\alpha,\beta}^{\gamma,\delta,q}((t_2-s)^q) - E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\right|^2 ds\right)^{1/2}$$
$$+ 2\left(\int_0^T \left|(t_2-s)^{q-1} - (t_1-s)^{q-1}\right|^2 ds\right)^{1/2}$$
$$\left(\int_0^T \left|E_{\alpha,\beta}^{\gamma,\delta,q}((t_1-s)^q)\right|^2 ds\right)^{1/2} M_g \tag{3.12}$$

Since the functions f and $E_{\alpha,\beta}^{\gamma,\delta,q}$ are continuous on compact $[0,T]\times[-r,r]$, $(t-s)^{1-\alpha}$ is continuous on compact $[0,T]\times[0,T]$, so uniformly continuous there. Hence, from (3.12) it follows that

$$|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. As a result $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Now the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded and equicontinuous, so it is compact in view of Arzelá-Ascoli theorem. By Lemma 3.1, $\{x_{\epsilon_n}\}$ converges uniformly to a function say $r \in C(J,\mathbb{R})$, i.e. $\lim_{n\to\infty} x_{\epsilon_n}(t) = r(t)$ uniformly on J. We show that the function r is a solution of the QFIE (1.1) on J. Now, $\{x_{\epsilon_n}\}$ is a solution of the QFIE

$$x_{\epsilon_n}(t) = \left[f_{\epsilon_n}(t, x_{\epsilon_n}(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ g_{\epsilon_n}(s, x_{\epsilon_n}(s))\, ds\right) \tag{3.13}$$

$$= \left[f(t, x_{\epsilon_n}(t)) + \epsilon_n\right]\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)$$
$$\times\left(g(s, x_{\epsilon_n}(s) + \epsilon_n\right) ds \tag{3.14}$$

for all $t \in J$. Now, taking the limit as by hypotheses $n \to \infty$ in the above inequality (3.13), we obtain

$$r(t) = \left[f(t, r(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, r(s))\, ds\right)$$

for all $t \in J$. This shows that r is a solution of the QFIE (1.1) defined on J. Finally, we shall show that $r(t)$ is the maximal solution of the QFIE (1.1) defined on J. To do this, let $x(t)$ be any solution of the QFIE(1.1) defined on J. Then,we have

$$x(t) = \left[f(t, x(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, x(s))\, ds\right) \tag{3.15}$$

for all $t \in J$. Similarly, if x_ϵ is any solution of the QFIE

$$x_\epsilon(t) = \left[f(t, x_\epsilon(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, x_\epsilon(s)) + \epsilon)\, ds \right) \tag{3.16}$$

then,

$$x_\epsilon(t) > \left[f(t, x_\epsilon(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, x_\epsilon(s)))\, ds \right) \tag{3.17}$$

for all $t \in J$. From the inequalities (3.15) and (3.17) it follows that $x(t) \leq x_\varepsilon(t),\ \ t\ J$. Taking the limit as $\epsilon \to 0$, we obtain $x(t) \leq r(t)$ for all $t \in J$. Hence r is a maximal solution of the QFIE (1.1) defined on J. In the same way minimal solution of the QFIE can be obtained. Further, we prove now that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to QFIE (1.1) on $J = [0, T]$.

Theorem 3.5. *Suppose that all the hypotheses of theorem 3.2 hold. Further, if there exists a function $u \in C(J, \mathbb{R})$ such that*

$$u(t) \leq \left[f(t, u(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, u(s)) + \epsilon)\, ds \right) \tag{3.18}$$

for all $t \in J$, then,

$$u(t) \leq r(t) \tag{3.19}$$

for all $t \in J$, where r is a maximal solution of the QFIE (1.1) on J.

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by theorem 3.2, $r_\epsilon(t)$ is a solution of the QFIE and that the limit

$$r(t) = \lim_{\epsilon \to 0} r_\epsilon(t) \tag{3.20}$$

is uniform on J and is a maximal solution of the QFIE(1.1) on J. Hence, we obtain

$$r_\epsilon(t) = \left[f(t, r_\epsilon(t)) + \epsilon\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)(g(s, r_\epsilon(s)) + \epsilon)\, ds \right) \tag{3.21}$$

for all $t \in J$. From the above inequality it follows that

$$r_\epsilon(t) > \left[f(t, r_\epsilon(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q)\ (g(s, r_\epsilon(s)))\, ds \right) \tag{3.22}$$

Now, we apply Lemma 3.3 to the inequalities (3.18) and (3.22) and conclude that

$$u(t) < r_\epsilon(t) \tag{3.23}$$

for all $t \in J$. This further in view of limit (3.20) implies that the inequality (3.19) holds on J. This completes the proof. Similarly, we have the following result for the QFIE (1.1) on J.

Theorem 3.6. *Suppose that all the hypotheses of theorem 3.2 hold. Further, if there exists a function $v \in C(J, \mathbb{R})$ such that*

$$v(t) \geq \left[f(t, v(t))\right] \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_{\alpha,\beta}^{\gamma,\delta,q}((t-s)^q) \; (g(s, v(s)))\, ds \right) \tag{3.24}$$

for all $t \in J$, then,

$$v(t) \geq \rho(t) \tag{3.25}$$

for all $t \in J$, where ρ is a minimal solution of the QFIE (1.1) on J.

4. Conclusion

Finally, using initial mixed conditions the quadratic fractional differential equation involving the generalized Mittag-Leffler function, the algorithm for the solution of this equation were develop using the sequence of successive approximation which converges monotonically. In addition from the same algorithm the existence of maximal and minimal solution were obtained which gives upper and lower bounds for the solution.

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